POLITECHNIKA RZESZOWSKA im. IGNACEGO ŁUKASIEWICZA

WYDZIAŁ ELEKTROTECHNIKI I INFORMATYKI POLITECHNIKI RZESZOWSKIEJ

Podstawy Elektromagnetyzmu

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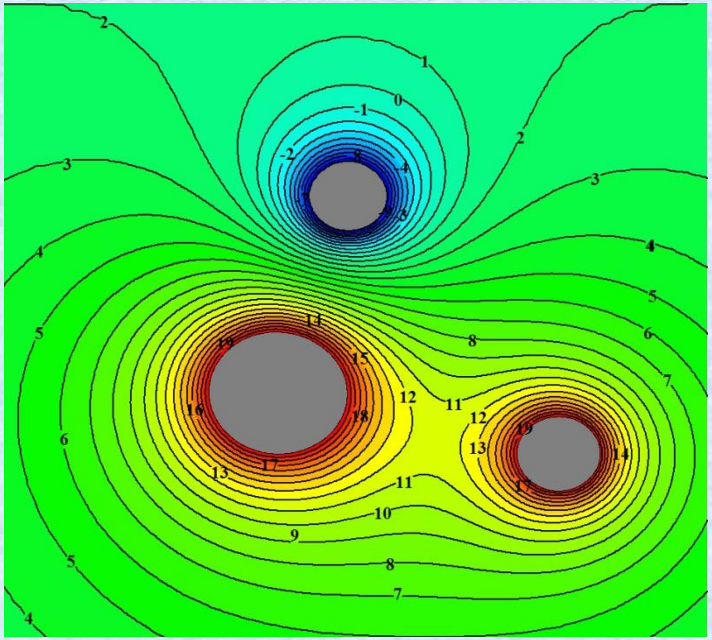
1.2. Operacje różniczkowe na polach

1.2.1. Pole skalarne

Pole skalarne

– funkcja odwzorowująca pewien obszar przestrzenny Ω na zbiór liczb rzeczywistych (ewentualnie zespolonych - pole zespolone)

$$\varphi = \varphi(\mathrm{P}), \qquad \mathrm{P} \in \Omega, \qquad \varphi \in \Re$$



We współrzędnych kartezjańskich:

$$\varphi = \varphi(x, y, z)$$

1.2.2 Pole wektorowe

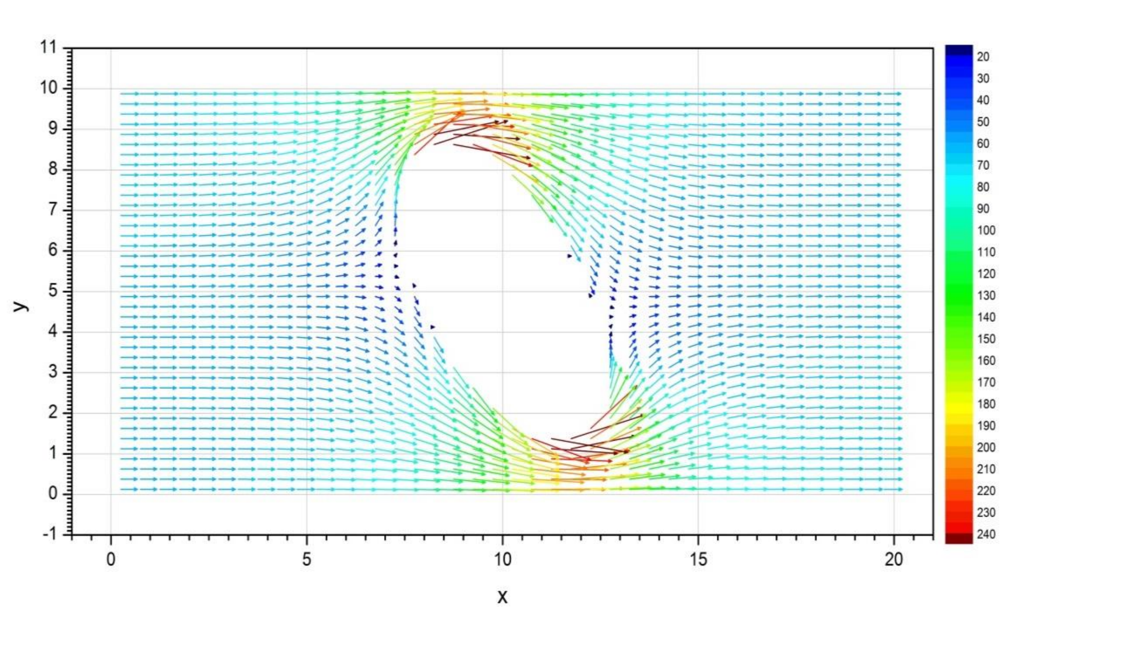
Pole wektorowe

– funkcja odwzorowująca pewien obszar przestrzenny Ω na zbiór wektorów

$$\vec{V} = \vec{V}(\mathrm{P}), \qquad \mathrm{P} \in \Omega$$

We współrzędnych kartezjańskich:

$$\vec{V} = \vec{V}(x, y, z) = \begin{bmatrix} V_x(x, y, z) \\ V_y(x, y, z) \\ V_z(x, y, z) \end{bmatrix}$$



1.2.3. Pochodne kierunkowe i cząstkowe

Pochodna kierunkowa

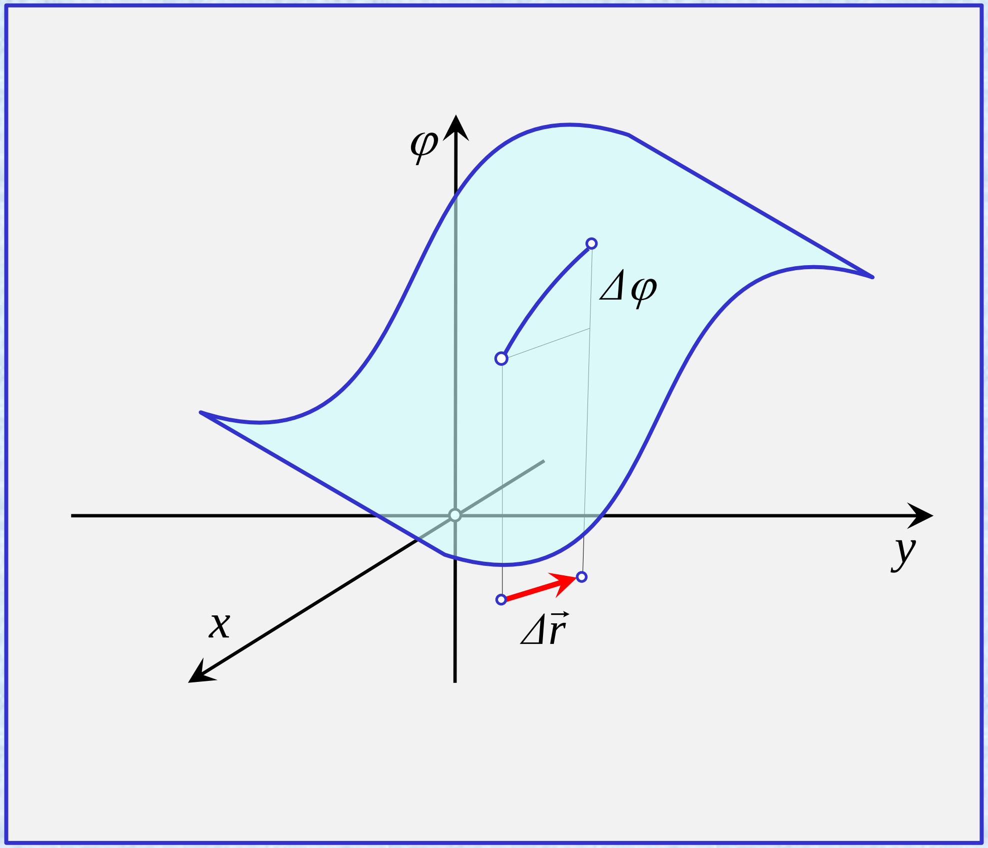
$$\frac{\partial \varphi}{\partial \vec{r}} = \lim_{\Delta r \to 0} \frac{\Delta \varphi}{\Delta r}$$

Pochodne cząstkowe

$$\Delta \vec{r} \parallel \mathrm{OX} \quad \Rightarrow \quad \frac{\partial \varphi}{\partial \vec{r}} \equiv \frac{\partial \varphi}{\partial x}$$

$$\Delta \vec{r} \parallel \mathrm{OY} \quad \Rightarrow \quad \frac{\partial \varphi}{\partial \vec{r}} \equiv \frac{\partial \varphi}{\partial y}$$

$$\Delta \vec{r} \parallel \mathrm{OZ} \quad \Rightarrow \quad \frac{\partial \varphi}{\partial \vec{r}} \equiv \frac{\partial \varphi}{\partial z}$$



1.2.4. Różniczka pola skalarnego

$$\mathrm{d}\varphi = \frac{\partial\varphi}{\partial x}\mathrm{d}x + \frac{\partial\varphi}{\partial y}\mathrm{d}y + \frac{\partial\varphi}{\partial z}\mathrm{d}z$$

Gradient

Wektorowy przyrost argumentu

$$\overrightarrow{\mathrm{grad}\varphi} = \left[\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right]$$

$$\overrightarrow{\mathrm{d}r} = [\mathrm{d}x, \mathrm{d}y, \mathrm{d}z]$$

czyli

$$\mathrm{d}\varphi = \left[\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}\right] \cdot [\mathrm{d}x,\ \mathrm{d}y,\ \mathrm{d}z] = \overrightarrow{\mathrm{grad}\varphi} \cdot \overrightarrow{\mathrm{d}r}$$

1.2.5. Gradient

$$\overrightarrow{\mathrm{grad}}\varphi = \left[\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right]$$

Właściwości

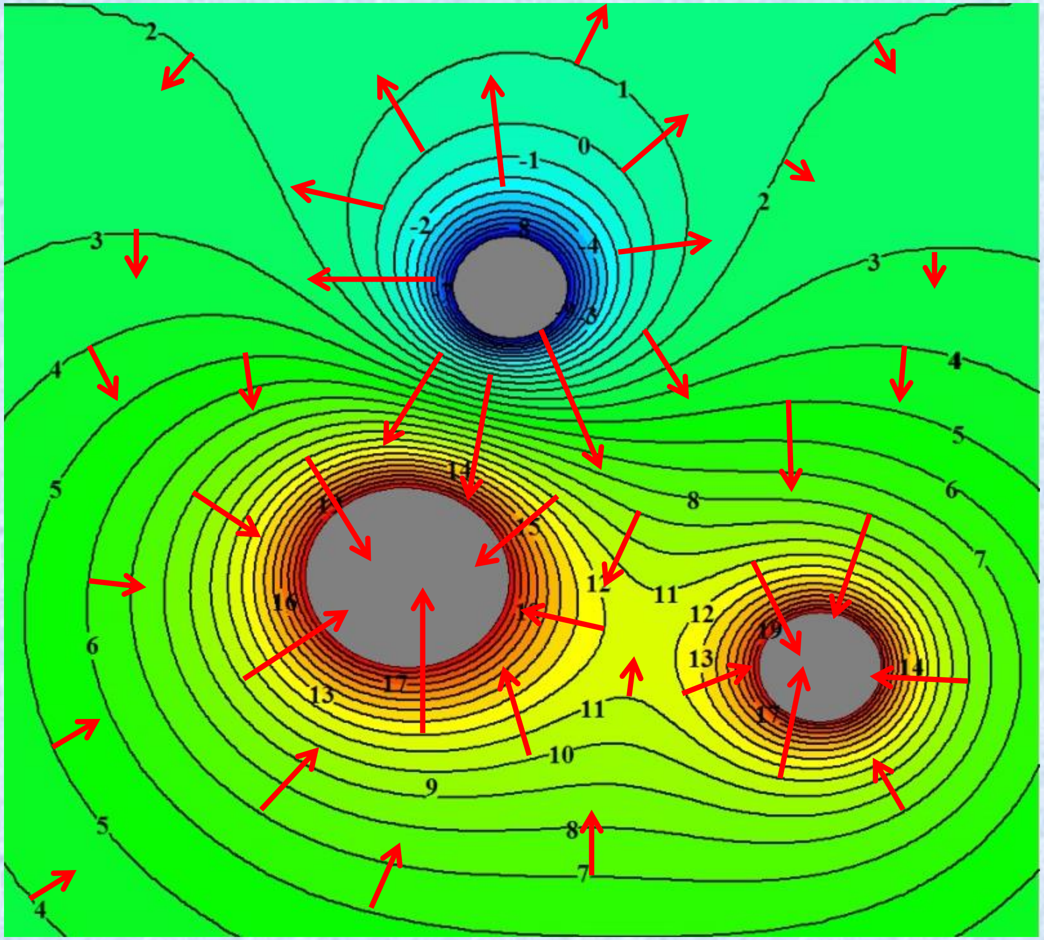
- gradient wskazuje kierunek najszybszego wzrostu pola
- wartość gradientu jest miarą szybkości narastania pola
- gradient jest prostopadły do powierzchni $\varphi = \mathrm{const}$ (powierzchnie ekwiskalarne)

Dowód:

$$\mathrm{d}\,\varphi = \overrightarrow{\mathrm{grad}}\varphi \cdot \overrightarrow{\mathrm{d}\,r}$$

$$\varphi = \mathrm{const} \quad \Rightarrow \quad \mathrm{d}\,\varphi = 0$$

$$\overrightarrow{\mathrm{grad}}\varphi \cdot \overrightarrow{\mathrm{d}\,r} = 0 \;\Rightarrow\; \overrightarrow{\mathrm{grad}}\varphi \perp \overrightarrow{\mathrm{d}\,r}$$



1.2.6. Operator nabla (Hamiltona)

$$\vec{\nabla} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right]$$

Działanie operatora nabla na pole skalarne

$$\vec{\nabla}\varphi = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right]\varphi = \left[\frac{\partial}{\partial x}\varphi, \frac{\partial}{\partial y}\varphi, \frac{\partial}{\partial z}\varphi \right] = \left[\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right] = \overrightarrow{\mathrm{grad}}\varphi$$

⇧

Analogia do mnożenia wektora przez skalar, ale:

$$\vec{\nabla}\varphi \neq \varphi\vec{\nabla} \quad !$$

„Mnożenie” operatora przez funkcję nie jest przemienne!

1.2.7. Działanie operatora nabla na pola wektorowe

Mnożenie skalarne – dywergencja (źródłowość, rozbieżność)

$$\operatorname{div}\vec{v} = \vec{\nabla}\cdot\vec{v} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]\cdot\left[v_x, \quad v_y, \quad v_z\right] = \frac{\partial}{\partial x}v_x + \frac{\partial}{\partial y}v_y + \frac{\partial}{\partial z}v_z = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Mnożenie wektorowe – rotacja (krążenie)

$$\operatorname{rot}\vec{v} = \vec{\nabla}\times\vec{v} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]\times\left[v_x, \quad v_y, \quad v_z\right] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \begin{bmatrix} \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \\ \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{bmatrix}$$

Przykłady

1. Obliczyć gradient pola skalarnego $\varphi(x,y,z)=\dfrac{2x+3z^2}{4y^3+z}$

$$\overrightarrow{\mathrm{grad}}\varphi=\vec{\nabla}\varphi=\left[\frac{\partial}{\partial x},\quad\frac{\partial}{\partial y},\quad\frac{\partial}{\partial z}\right]\varphi=\left[\frac{\partial\varphi}{\partial x},\quad\frac{\partial\varphi}{\partial y},\quad\frac{\partial\varphi}{\partial z}\right]$$

$$\frac{\partial\varphi}{\partial x}=\frac{\partial}{\partial x}\frac{2x+3z^2}{4y^3+z}=\frac{1}{4y^3+z}\frac{\partial}{\partial x}\left(2x+3z^2\right)=\frac{2}{4y^3+z}$$

$$\frac{\partial\varphi}{\partial y}=\frac{\partial}{\partial y}\frac{2x+3z^2}{4y^3+z}=\left(2x+3z^2\right)\frac{\partial}{\partial y}\frac{1}{4y^3+z}=\left(2x+3z^2\right)\left(-\frac{1}{\left(4y^3+z\right)^2}\right)\frac{\partial}{\partial y}\left(4y^3+z\right)=-12y^2\frac{2x+3z^2}{\left(4y^3+z\right)^2}$$

$$\frac{\partial\varphi}{\partial z}=\frac{\partial}{\partial z}\frac{2x+3z^2}{4y^3+z}=\frac{\left(4y^3+z\right)\frac{\partial}{\partial z}\left(2x+3z^2\right)-\left(2x+3z^2\right)\frac{\partial}{\partial z}\left(4y^3+z\right)}{\left(4y^3+z\right)^2}=\frac{6z\left(4y^3+z\right)-1\left(2x+3z^2\right)}{\left(4y^3+z\right)^2}=\frac{-2x+24y^3z+3z^2}{\left(4y^3+z\right)^2}$$

$$\overrightarrow{\mathrm{grad}}\varphi=\left[\frac{2}{4y^3+z},\quad-12y^2\frac{2x+3z^2}{\left(4y^3+z\right)^2},\quad\frac{-2x+24y^3z+3z^2}{\left(4y^3+z\right)^2}\right]$$

Przykłady

2. Obliczyć dywergencję pola wektorowego $\vec{V} = \left[\frac{4xy}{z}, \quad 3xe^{-yz}\cos(2y), \quad 2y\ln(5xz) \right]$

$$\operatorname{div}\vec{V} = \vec{\nabla}\cdot\vec{V} = \left[\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z} \right]\cdot\left[V_x, \quad V_y, \quad V_z \right] = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$\frac{\partial V_x}{\partial x} = \frac{\partial}{\partial x}\frac{4xy}{z} = \frac{4y}{z}$$

$$\frac{\partial V_y}{\partial y} = \frac{\partial}{\partial y}3x\mathrm{e}^{-yz}\cos(2y) = 3x\left(\cos(2y)\frac{\partial}{\partial y}\mathrm{e}^{-yz} + \mathrm{e}^{-yz}\frac{\partial}{\partial y}\cos(2y) \right) = 3x\left(-z\cos(2y)\mathrm{e}^{-yz} - 2\mathrm{e}^{-yz}\sin(2y)\right) =$$

$$= -3x\mathrm{e}^{-yz}\left(z\cos(2y) + 2\sin(2y)\right)$$

$$\frac{\partial V_z}{\partial z} = \frac{\partial}{\partial z}2y\ln(5xz) = 2y\frac{1}{5xz}5x = \frac{2y}{z}$$

$$\operatorname{div}\vec{V} = \frac{6y}{z} - 3x\mathrm{e}^{-yz}\left(z\cos(2y) + 2\sin(2y)\right)$$

Przykłady

3. Obliczyć rotację pola wektorowego $\vec{V} = \left[\frac{4xy}{z}, \quad 3xe^{-yz}\cos(2y), \quad 2y\ln(5xz) \right]$

$$\overrightarrow{\mathrm{rot}}\vec{V} = \vec{\nabla} \times \vec{V} = \left[\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z} \right] \times \left[V_x, \quad V_y, \quad V_z \right] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \begin{bmatrix} \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \\ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \\ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{bmatrix}$$

$$\frac{\partial V_z}{\partial y} = \frac{\partial}{\partial y} 2y\ln(5xz) = 2\ln(5xz)$$

$$\frac{\partial V_y}{\partial z} = \frac{\partial}{\partial z} 3xe^{-yz}\cos(2y) = -3xye^{-yz}\cos(2y)$$

$$\frac{\partial V_x}{\partial z} = \frac{\partial}{\partial z}\frac{4xy}{z} = -\frac{4xy}{z^2}$$

$$\frac{\partial V_z}{\partial x} = \frac{\partial}{\partial x} 2y\ln(5xz) = \frac{2y}{x}$$

$$\frac{\partial V_y}{\partial x} = \frac{\partial}{\partial x} 3xe^{-yz}\cos(2y) = 3e^{-yz}\cos(2y)$$

$$\frac{\partial V_x}{\partial y} = \frac{\partial}{\partial y}\frac{4xy}{z} = \frac{4x}{z}$$

$$\overrightarrow{\mathrm{rot}}\vec{V} = \begin{bmatrix} 2\ln(5xz) - \frac{\partial V_y}{\partial z} \\ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \\ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{bmatrix}$$

Przykłady

4. Wykazać, że dla dowolnych pól $\varphi, \vec{V}$ zachodzi tożsamość:

$$\mathrm{div}\left(\varphi\vec{V}\right) = \vec{V} \cdot \overrightarrow{\mathrm{grad}}\varphi + \varphi\,\mathrm{div}\vec{V}$$

$$\mathrm{div}\left(\varphi\vec{V}\right) = \mathrm{div}\left(\varphi\left[V_x, \quad V_y, \quad V_z\right]\right) = \mathrm{div}\left(\left[\varphi V_x, \quad \varphi V_y, \quad \varphi V_z\right]\right) = \frac{\partial}{\partial x}\left(\varphi V_x\right) + \frac{\partial}{\partial y}\left(\varphi V_y\right) + \frac{\partial}{\partial z}\left(\varphi V_z\right) =$$

$$= V_x \frac{\partial \varphi}{\partial x} + \varphi \frac{\partial V_x}{\partial x} + \ldots$$

Przykłady

2. Obliczyć dywergencję pola wektorowego $\vec{V} = \left[\dfrac{4xy}{z}, \quad 3xe^{-yz}\cos(2y), \quad 2y\ln(5xz) \right]$

Rozwiązanie

$$\mathrm{div}\vec{V} = \vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$\frac{\partial V_x}{\partial x} = \frac{\partial}{\partial x}\frac{4xy}{z} = \frac{4y}{z}$$

$$\frac{\partial V_y}{\partial y} = \frac{\partial}{\partial y}\left(3xe^{-yz}\cos(2y)\right) = 3xe^{-yz}(-z)\cos(2y) + 3xe^{-yz}(-\sin(2y))\cdot 2 =$$

$$= -3xze^{-yz}\cos(2y) - 6xe^{-yz}\sin(2y) = -3xe^{-yz}\left(z\cos(2y) + 2\sin(2y)\right)$$

$$\frac{\partial V_z}{\partial z} = \frac{\partial}{\partial z}\left(2y\ln(5xz)\right) = \frac{2y}{5xz}5x = \frac{2y}{z}$$

Odpowiedź

$$\mathrm{div}\vec{V} = \frac{6y}{z} - 3xe^{-yz}\left(z\cos(2y) + 2\sin(2y)\right)$$

3. Obliczyć rotację pola wektorowego $\vec{V} = \left[\frac{4xy}{z}, \quad 3xe^{-yz}\cos(2y), \quad 2y\ln(5xz) \right]$

Rozwiązanie

$$\overrightarrow{\mathrm{rot}}\vec{V} = \vec{\nabla} \times \vec{V} = \left[\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z} \right] \times \left[V_x, \quad V_y, \quad V_z \right] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \begin{bmatrix} \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \\ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \\ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{bmatrix}$$

$$\frac{\partial V_z}{\partial y} = \frac{\partial}{\partial y} 2y\ln(5xz) = 2\ln(5xz)$$

$$\frac{\partial V_y}{\partial z} = \frac{\partial}{\partial z} 3xe^{-yz}\cos(2y) = -3xye^{-yz}\cos(2y)$$

$$\frac{\partial V_x}{\partial z} = \frac{\partial}{\partial z}\frac{4xy}{z} = -\frac{4xy}{z^2}$$

$$\frac{\partial V_z}{\partial x} = \frac{\partial}{\partial x} 2y\ln(5xz) = 2y\frac{5x}{5xz} = \frac{2y}{z}$$

$$\frac{\partial V_y}{\partial x} = \frac{\partial}{\partial x} 3xe^{-yz}\cos(2y) = -3ye^{-yz}\cos(2y)$$

$$\frac{\partial V_x}{\partial y} = \frac{\partial}{\partial y}\frac{4xy}{z} = \frac{4x}{z}$$

$$\overrightarrow{\mathrm{rot}}\vec{V} = \begin{bmatrix} 2\ln(5xz) + 3xye^{-yz}\cos(2y) \\ -\frac{4xy}{z^2} - \frac{2y}{z} \\ -3ye^{-yz}\cos(2y) - \frac{4x}{z} \end{bmatrix}$$

1.2.8. Operacje różniczkowe drugiego rzędu

Dywergencja z gradientu (laplasjan)

$$\operatorname{div}\left(\overrightarrow{\operatorname{grad}}\varphi\right)=\vec{\nabla}\cdot\left(\vec{\nabla}\varphi\right)=\left[\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right]\cdot\left[\frac{\partial\varphi}{\partial x},\ \frac{\partial\varphi}{\partial y},\ \frac{\partial\varphi}{\partial z}\right]=$$

$$=\frac{\partial}{\partial x}\frac{\partial\varphi}{\partial x}+\frac{\partial}{\partial y}\frac{\partial\varphi}{\partial y}+\frac{\partial}{\partial z}\frac{\partial\varphi}{\partial z}=\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}$$

<u>laplasjan</u>

$$\Delta\equiv\vec{\nabla}^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

(operator skalarny!)

czyli: $$\operatorname{div}\left(\overrightarrow{\operatorname{grad}}\varphi\right)=\Delta\varphi$$

1.2.8. Operacje różniczkowe drugiego rzędu

Laplasjan pola wektorowego

$$\Delta\vec{v} = \begin{bmatrix} \Delta v_x \\ \Delta v_y \\ \Delta v_z \end{bmatrix} = \begin{bmatrix} \dfrac{\partial^2 v_x}{\partial x^2} + \dfrac{\partial^2 v_x}{\partial y^2} + \dfrac{\partial^2 v_x}{\partial z^2} \\ \dfrac{\partial^2 v_y}{\partial x^2} + \dfrac{\partial^2 v_y}{\partial y^2} + \dfrac{\partial^2 v_y}{\partial z^2} \\ \dfrac{\partial^2 v_z}{\partial x^2} + \dfrac{\partial^2 v_z}{\partial y^2} + \dfrac{\partial^2 v_z}{\partial z^2} \end{bmatrix}$$

Analogicznnie do mnożenia skalara przez wektor

1.2.8. Operacje różniczkowe drugiego rzędu

Rotacja z gradientu

$$\overrightarrow{\mathrm{rot}}\left(\overrightarrow{\mathrm{grad}}\varphi\right)=\vec{\nabla}\times\left(\vec{\nabla}\varphi\right)=\left[\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right]\times\left[\frac{\partial\varphi}{\partial x},\ \frac{\partial\varphi}{\partial y},\ \frac{\partial\varphi}{\partial z}\right]=$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z}\end{vmatrix}=\begin{bmatrix}\frac{\partial}{\partial y}\frac{\partial\varphi}{\partial z}-\frac{\partial}{\partial z}\frac{\partial\varphi}{\partial y}\\ \frac{\partial}{\partial z}\frac{\partial\varphi}{\partial x}-\frac{\partial}{\partial x}\frac{\partial\varphi}{\partial z}\\ \frac{\partial}{\partial x}\frac{\partial\varphi}{\partial y}-\frac{\partial}{\partial y}\frac{\partial\varphi}{\partial x}\end{bmatrix}=\begin{bmatrix}\frac{\partial^2\varphi}{\partial y\partial z}-\frac{\partial^2\varphi}{\partial z\partial y}\\ \frac{\partial^2\varphi}{\partial z\partial x}-\frac{\partial^2\varphi}{\partial x\partial z}\\ \frac{\partial^2\varphi}{\partial x\partial y}-\frac{\partial^2\varphi}{\partial y\partial x}\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}=\vec{0}$$

1.2.8. Operacje różniczkowe drugiego rzędu

Dywergencja z rotacji

$$\operatorname{div}\left(\overrightarrow{\operatorname{rot}\vec{v}}\right) = \vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{v}\right) = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]\cdot\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_x & v_y & v_z\end{vmatrix} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]\cdot\begin{bmatrix}\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\\ \frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\\ \frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\end{bmatrix} =$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right) =$$

$$= \frac{\partial^2 v_z}{\partial x\partial y} - \frac{\partial^2 v_y}{\partial x\partial z} + \frac{\partial^2 v_x}{\partial y\partial z} - \frac{\partial^2 v_z}{\partial y\partial x} + \frac{\partial^2 v_y}{\partial z\partial x} - \frac{\partial^2 v_x}{\partial z\partial y} = 0$$

1.2.8. Operacje różniczkowe drugiego rzędu

Rotacja z rotacji

$$\overrightarrow{\mathrm{rot}}\left(\overrightarrow{\mathrm{rot}}\vec{v}\right) = \overrightarrow{\mathrm{grad}}\left(\mathrm{div}\,\vec{v}\right) - \Delta\vec{v}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{v}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{v}\right) - \Delta\vec{v}$$

Operacje różniczkowe na polach - podsumowanie

Operator	Operacje I rzędu	Operacje II rzędu
nabla: $\vec{\nabla} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$	$\overrightarrow{\mathrm{grad}}\varphi = \vec{\nabla}\varphi$	$\mathrm{div}\left(\overrightarrow{\mathrm{grad}}\varphi\right) = \Delta\varphi$
	$\mathrm{div}\,\vec{v} = \vec{\nabla}\cdot\vec{v}$	$\overrightarrow{\mathrm{rot}}\left(\overrightarrow{\mathrm{grad}}\varphi\right) = \vec{0}$
laplasjan: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$	$\overrightarrow{\mathrm{rot}}\vec{v} = \vec{\nabla}\times\vec{v}$	$\mathrm{div}\left(\overrightarrow{\mathrm{rot}}\vec{v}\right) = 0$
		$\overrightarrow{\mathrm{rot}}\left(\overrightarrow{\mathrm{rot}}\vec{v}\right) = \overrightarrow{\mathrm{grad}}\left(\mathrm{div}\,\vec{v}\right) - \Delta\vec{v}$

Przykłady

1. Obliczyć gradient pola skalarnego $\varphi(x,y,z)=\dfrac{2x+3z^2}{4y^3+z}$

Rozwiązanie

$$\overrightarrow{\mathrm{grad}}\varphi=\vec{\nabla}\varphi=\left[\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}\right]\varphi=\left[\frac{\partial\varphi}{\partial x},\ \frac{\partial\varphi}{\partial y},\ \frac{\partial\varphi}{\partial z}\right]$$

$$\frac{\partial\varphi}{\partial x}=\frac{\partial}{\partial x}\frac{2x+3z^2}{4y^3+z}=\frac{\partial}{\partial x}\left(\frac{2x}{4y^3+z}+\frac{3z^2}{4y^3+z}\right)=\frac{2}{4y^3+z}$$

$$\frac{\partial\varphi}{\partial y}=\frac{\partial}{\partial y}\left(\frac{2x+3z^2}{4y^3+z}\right)=-\frac{2x+3z^2}{\left(4y^3+z\right)^2}\cdot 12y^2$$

$$\frac{\partial\varphi}{\partial z}=\frac{\partial}{\partial z}\left(\frac{2x+3z^2}{4y^3+z}\right)=\frac{6z\left(4y^3+z\right)-1\left(2x+3z^2\right)}{\left(4y^3+z\right)^2}=\frac{-2x+3z^2+24y^3z}{\left(4y^3+z\right)^2}$$

Odpowiedź

$$\overrightarrow{\mathrm{grad}}\varphi=\left[\frac{2}{4y^3+z},\ -12y^2\frac{2x+3z^2}{\left(4y^3+z\right)^2},\ \frac{-2x+3z^2+24y^3z}{\left(4y^3+z\right)^2}\right]$$

Przykłady

2. Obliczyć dywergencję pola wektorowego $\vec{V} = \left[\dfrac{4xy}{z}, \quad 3xe^{-yz}\cos(2y), \quad 2y\ln(5xz) \right]$

Rozwiązanie

$$\operatorname{div}\vec{V} = \vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$\frac{\partial V_x}{\partial x} = \frac{\partial}{\partial x}\frac{4xy}{z} = \frac{4y}{z}$$

$$\frac{\partial V_y}{\partial y} = \frac{\partial}{\partial y}\left(3xe^{-yz}\cos(2y)\right) = 3xe^{-yz}(-z)\cos(2y) + 3xe^{-yz}(-\sin(2y))\cdot 2 =$$

$$= -3xze^{-yz}\cos(2y) - 6xe^{-yz}\sin(2y) = -3xe^{-yz}\left(z\cos(2y) + 2\sin(2y)\right)$$

$$\frac{\partial V_z}{\partial z} = \frac{\partial}{\partial z}\left(2y\ln(5xz)\right) = \frac{2y}{5xz}5x = \frac{2y}{z}$$

Odpowiedź

$$\operatorname{div}\vec{V} = \frac{6y}{z} - 3xe^{-yz}\left(z\cos(2y) + 2\sin(2y)\right)$$

3. Obliczyć rotację pola wektorowego $\vec{V} = \left[\frac{4xy}{z}, \quad 3xe^{-yz}\cos(2y), \quad 2y\ln(5xz) \right]$

Rozwiązanie

$$\overrightarrow{\mathrm{rot}}\vec{V} = \vec{\nabla} \times \vec{V} = \left[\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z} \right] \times \left[V_x, \quad V_y, \quad V_z \right] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \begin{bmatrix} \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \\ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \\ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{bmatrix}$$

$$\frac{\partial V_z}{\partial y} = \frac{\partial}{\partial y} 2y\ln(5xz) = 2\ln(5xz)$$

$$\frac{\partial V_y}{\partial z} = \frac{\partial}{\partial z} 3xe^{-yz}\cos(2y) = -3xye^{-yz}\cos(2y)$$

$$\frac{\partial V_x}{\partial z} = \frac{\partial}{\partial z}\frac{4xy}{z} = -\frac{4xy}{z^2}$$

$$\frac{\partial V_z}{\partial x} = \frac{\partial}{\partial x} 2y\ln(5xz) = 2y\frac{5x}{5xz} = \frac{2y}{z}$$

$$\frac{\partial V_y}{\partial x} = \frac{\partial}{\partial x} 3xe^{-yz}\cos(2y) = -3ye^{-yz}\cos(2y)$$

$$\frac{\partial V_x}{\partial y} = \frac{\partial}{\partial y}\frac{4xy}{z} = \frac{4x}{z}$$

$$\overrightarrow{\mathrm{rot}}\vec{V} = \begin{bmatrix} 2\ln(5xz) + 3xye^{-yz}\cos(2y) \\ -\frac{4xy}{z^2} - \frac{2y}{z} \\ -3ye^{-yz}\cos(2y) - \frac{4x}{z} \end{bmatrix}$$

Przykłady

4. Wykazać, że dla dowolnych pól $\varphi, \vec{V}$ zachodzi tożsamość:

$$\mathrm{div}\left(\varphi\vec{V}\right) = \vec{V}\cdot\overrightarrow{\mathrm{grad}}\varphi + \varphi\,\mathrm{div}\vec{V}$$

Rozwiązanie

Tu korzystamy ze wzoru na pochodną iloczynu (φ nie jest stałą, więc nie można prosto wyciągać przed znak pochodnej!)

Rozpisujemy lewą stronę:

$$\mathrm{L} = \mathrm{div}\left(\varphi\vec{V}\right) = \vec{\nabla}\cdot\left[\varphi V_x,\quad \varphi V_y,\quad \varphi V_z\right] = \frac{\partial}{\partial x}\left(\varphi V_x\right) + \frac{\partial}{\partial y}\left(\varphi V_y\right) + \frac{\partial}{\partial z}\left(\varphi V_z\right) =$$

$$= \frac{\partial\varphi}{\partial x}V_x + \varphi\frac{\partial V_x}{\partial x} + \frac{\partial\varphi}{\partial y}V_y + \varphi\frac{\partial V_y}{\partial y} + \frac{\partial\varphi}{\partial z}V_z + \varphi\frac{\partial V_z}{\partial z}$$

Rozpisujemy prawą stronę:

$$\mathrm{P} = \vec{V}\cdot\overrightarrow{\mathrm{grad}}\varphi + \varphi\,\mathrm{div}\vec{V} = \left[V_x,\quad V_y,\quad V_z\right]\cdot\left[\frac{\partial\varphi}{\partial x},\quad \frac{\partial\varphi}{\partial y},\quad \frac{\partial\varphi}{\partial z}\right] + \varphi\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) =$$

$$= V_x\frac{\partial\varphi}{\partial x} + V_y\frac{\partial\varphi}{\partial y} + V_z\frac{\partial\varphi}{\partial z} + \varphi\frac{\partial V_x}{\partial x} + \varphi\frac{\partial V_y}{\partial y} + \varphi\frac{\partial V_z}{\partial z}$$

Stwierdzamy, że L = P

Zadania do samodzielnego przerobienia

Dane są pola:

skalarne $\varphi = \dfrac{4z - x^3}{y + z^2} - 3xy\ln(2yz)$

i wektorowe $\vec{V} = \left[4x\mathrm{e}^{3yz}, \quad \dfrac{x + 2z}{3y}, \quad 4x\sin\left(\dfrac{z}{2y}\right) \right]$

1. Obliczyć gradient pola φ
2. Obliczyć dywergencję pola $\vec{V}$
3. Obliczyć rotację pola $\vec{V}$
4. Wykazać, że dla dowolnych pól $\varphi, \vec{V}$ zachodzi tożsamość:

$$\overrightarrow{\mathrm{rot}}\left(\varphi\vec{V}\right) = \overrightarrow{\mathrm{grad}}\,\varphi \times \vec{V} + \varphi\,\overrightarrow{\mathrm{rot}}\vec{V}$$