Zadanie. Rozwiązać układ równań: $\underline{A}\,\underline{x} = \underline{L}$, stosując trzy metody wyznaczenia pseudoodwrotności macierzy. Macierz $\underline{A}$ jest macierzą prostokątną pionową.

$$\underline{A} = \begin{bmatrix} -1 & 1 & 0 & \vdots & 0 \\ 0 & -1 & 1 & \vdots & 0 \\ 0 & 0 & -1 & \vdots & 1 \\ 1 & 0 & 0 & \vdots & -1 \\ -1 & 0 & 1 & \vdots & 0 \end{bmatrix}; \; \underline{L} = \begin{bmatrix} 16 \\ -8 \\ 8 \\ -16 \\ 0 \end{bmatrix} \qquad r\{\underline{A}\} = 3, \quad d = 1 \text{ - macierz nieregularna}$$

Rozwiązanie : $\hat{\underline{x}} = \underline{A}^{+}\underline{L}$ **(** $\underline{A}^{+}$ **- pseudoodwrotność)**

Metoda 1 - z zastosowaniem odwrotności normalnej.

$$\underline{A}^{+} = \underline{N}^{*}\underline{A}^{T}$$

$$\underline{N}^{*} = \underline{N}\left(\underline{N}\,\underline{N}\right)_{o}^{-1}$$

$$\underline{N} = \underline{A}^{T}\underline{A}$$

$$\underline{N} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \quad (r\{\underline{N}\} = 3 \quad d = 1)$$

$$\left(\underline{N}\underline{N}\right)_{o} = \begin{bmatrix} 12 & -4 & -4 & \vdots & \\ -4 & 6 & -4 & \vdots & \\ -4 & -4 & 12 & \vdots & \\ \hdashline & & & \vdots & \end{bmatrix} = 2 \times \begin{bmatrix} 6 & -2 & -2 & \vdots & \\ -2 & 3 & -2 & \vdots & \\ -2 & -2 & 6 & \vdots & \\ \hdashline & & & \vdots & \end{bmatrix}$$

$$\left(\underline{N}\underline{N}\right)_{o}^{-1} = \frac{1}{2 \times 32}\begin{bmatrix} 14 & 16 & 10 & \vdots & 0 \\ 16 & 32 & 16 & \vdots & 0 \\ 10 & 16 & 14 & \vdots & 0 \\ \hdashline 0 & 0 & 0 & \vdots & 0 \end{bmatrix} = \frac{1}{32}\begin{bmatrix} 7 & 8 & 5 & \vdots & 0 \\ 8 & 16 & 8 & \vdots & 0 \\ 5 & 8 & 7 & \vdots & 0 \\ \hdashline 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

$$\underline{N}^{*} = \underline{N}\left(\underline{N}\,\underline{N}\right)_{o}^{-1} = \frac{1}{32}\begin{bmatrix} 8 & 0 & 0 & 0 \\ 4 & 16 & 4 & 0 \\ 0 & 0 & 8 & 0 \\ -12 & -16 & -12 & 0 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ -3 & -4 & -3 & 0 \end{bmatrix}$$

$$\underline{A}^{+} = \underline{N}^{*}\underline{A}^{T} = \frac{1}{8}\begin{bmatrix} -2 & 0 & 0 & 2 & -2 \\ 3 & -3 & -1 & 1 & 0 \\ 0 & 2 & -2 & 0 & 2 \\ -1 & 1 & 3 & -3 & 0 \end{bmatrix}$$

Metoda 2 - Helmerta - Wolfa.

$$\underline{A}^{+} = \begin{bmatrix} \underline{N}_1 \\ \underline{N}_2^{T} \end{bmatrix} \left(\underline{N}_1 \underline{N}_1 + \underline{N}_2 \underline{N}_2^{T} \right)^{-1} \underline{A}_1^{T}$$

przy czym $\underline{N}_1 = \underline{A}_1^{T} \underline{A}_1$, $\underline{N}_2 = \underline{A}_1^{T} \underline{A}_2$

$$\underline{A}_1 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad ; \quad \underline{A}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\underline{N}_1 = \begin{bmatrix} -1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \end{bmatrix} \times \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 3 & -1 & -1 & 1 \vee \\ -1 & 2 & -1 & 0 \vee \\ -1 & -1 & 3 & 1 \vee \end{array}\right]$$

$$\underline{N}_2 = \begin{bmatrix} -1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

$$\underline{N}_1 \underline{N}_1 = \begin{bmatrix} 11 & -4 & -5 \\ -4 & 6 & -4 \\ -5 & -4 & 11 \end{bmatrix} \quad ; \quad \underline{N}_2 \underline{N}_2^{T} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\left(\underline{N}_1 \underline{N}_1 + \underline{N}_2 \underline{N}_2^{T} \right) = \begin{bmatrix} 12 & -4 & -4 \\ -4 & 6 & -4 \\ -4 & -4 & 12 \end{bmatrix} \qquad \left(\underline{N}_1 \underline{N}_1 + \underline{N}_2 \underline{N}_2^{T} \right)^{-1} = \frac{1}{256} \begin{bmatrix} 56 & 64 & 40 \\ 64 & 128 & 64 \\ 40 & 64 & 56 \end{bmatrix} = \frac{1}{32} \begin{bmatrix} 7 & 8 & 5 \\ 8 & 16 & 8 \\ 5 & 8 & 7 \end{bmatrix}$$

$$\begin{bmatrix} \underline{N}_1 \\ \underline{N}_2^{T} \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \\ -1 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \underline{N}_1 \\ \underline{N}_2^{T} \end{bmatrix} \left(\underline{N}_1 \underline{N}_1 + \underline{N}_2 \underline{N}_2^{T} \right)^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \\ -1 & 0 & -1 \end{bmatrix} \times \frac{1}{32} \begin{bmatrix} 7 & 8 & 5 \\ 8 & 16 & 8 \\ 5 & 8 & 7 \end{bmatrix} = \frac{1}{32} \begin{bmatrix} 8 & 0 & 0 \\ 4 & 16 & 4 \\ 0 & 0 & 8 \\ -12 & -16 & -12 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 1 \\ 0 & 0 & 2 \\ -3 & -4 & -3 \end{bmatrix}$$

$$\mathbf{A}^{+} = \frac{1}{8} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 1 \\ 0 & 0 & 2 \\ -3 & -4 & -3 \end{bmatrix} \times \begin{bmatrix} -1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -2 & 0 & 0 & 2 & -2 \\ 3 & -3 & -1 & 1 & 0 \\ 0 & 2 & -2 & 0 & 2 \\ -1 & 1 & 3 & -3 & 0 \end{bmatrix}$$

Metoda 3 - z zastosowaniem wektorów własnych.

$$\underline{A}^{+} = \left(\underline{A}^{T}\underline{A}\right)^{+}\underline{A}^{T} = \underline{N}^{+}\underline{A}^{T}$$

$$\underline{N}^{+} = \left\{\underline{N} + \underline{S}_o\,\underline{S}_o^{\,T}\right\}^{-1} - \underline{S}_o\underline{S}_o^{\,T}$$

$\underline{S}_o$ - macierz modalna dla $\lambda = 0$ (macierz utworzona z wektorów własnych przyporządkowanych zerowym wartościom własnym macierzy $\underline{N}$)

$$\underline{N} = \underline{A}^{T}\underline{A} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Macierz charakterystycza $\Leftarrow$ $\left(\underline{N} - \lambda E\right)$

Równanie charakterystyczne $\Leftarrow$ $\det\left\{\underline{N} - \lambda E\right\} = 0$ $\rightarrow$ wartości własne

Macierz spektralna $\underline{D}_{\lambda} = \mathrm{diag}\left(0,2,4,4\right)$

$$\underline{S}_o = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

$$\underline{S}_o\underline{S}_o^{\,T} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} \times \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$$

$$\underline{N} + \underline{S}_o\underline{S}_o^{\,T} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix} = \begin{bmatrix} 3.25 & -0.75 & -0.75 & -0.75 \\ -0.75 & 2.25 & -0.75 & 0.25 \\ -0.75 & -0.75 & 3.25 & -0.75 \\ -0.75 & 0.25 & -0.75 & 2.25 \end{bmatrix}$$

$$\left(\underline{N} + \underline{S}_o\underline{S}_o^{\,T}\right)^{-1} = \begin{bmatrix} 0.4375 & 0.1875 & 0.1875 & 0.1875 \\ 0.1875 & 0.5625 & 0.1875 & 0.0625 \\ 0.1875 & 0.1875 & 0.4375 & 0.1875 \\ 0.1875 & 0.0625 & 0.1875 & 0.5625 \end{bmatrix}$$

$$\underline{N}^{+} = \begin{bmatrix} 0.4375 & 0.1875 & 0.1875 & 0.1875 \\ 0.1875 & 0.5625 & 0.1875 & 0.0625 \\ 0.1875 & 0.1875 & 0.4375 & 0.1875 \\ 0.1875 & 0.0625 & 0.1875 & 0.5625 \end{bmatrix} - \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix} = \begin{bmatrix} 0.1875 & -0.0625 & -0.0625 & -0.0625 \\ -0.0625 & 0.3125 & -0.0625 & -0.1875 \\ -0.0625 & -0.0625 & 0.1875 & -0.0625 \\ -0.0625 & -0.1875 & -0.0625 & 0.3125 \end{bmatrix}$$

$$\underline{A}^{+} = \underline{N}^{+}\underline{A}^{T} = \begin{bmatrix} 0.1875 & -0.0625 & -0.0625 & -0.0625 \\ -0.0625 & 0.3125 & -0.0625 & -0.1875 \\ -0.0625 & -0.0625 & 0.1875 & -0.0625 \\ -0.0625 & -0.1875 & -0.0625 & 0.3125 \end{bmatrix} \times \begin{bmatrix} -1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -2 & 0 & 0 & 2 & -2 \\ 3 & -3 & -1 & 1 & 0 \\ 0 & 2 & -2 & 0 & 2 \\ -1 & 1 & 3 & -3 & 0 \end{bmatrix}$$

Zastosowane metody wyznaczenia pseudoodwrotności przyniosły identyczne wyniki, stąd:

$$\underline{\hat{x}} = \underline{A}^{+}\underline{L} = \frac{1}{8}\begin{bmatrix} -2 & 0 & 0 & 2 & -2 \\ 3 & -3 & -1 & 1 & 0 \\ 0 & 2 & -2 & 0 & 2 \\ -1 & 1 & 3 & -3 & 0 \end{bmatrix} \times \begin{bmatrix} 16 \\ -8 \\ 8 \\ -16 \\ 0 \end{bmatrix} = \begin{bmatrix} -8.00 \\ 6.00 \\ -4.00 \\ 6.00 \end{bmatrix}$$